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Tensor Product Representations of Regular Transductions

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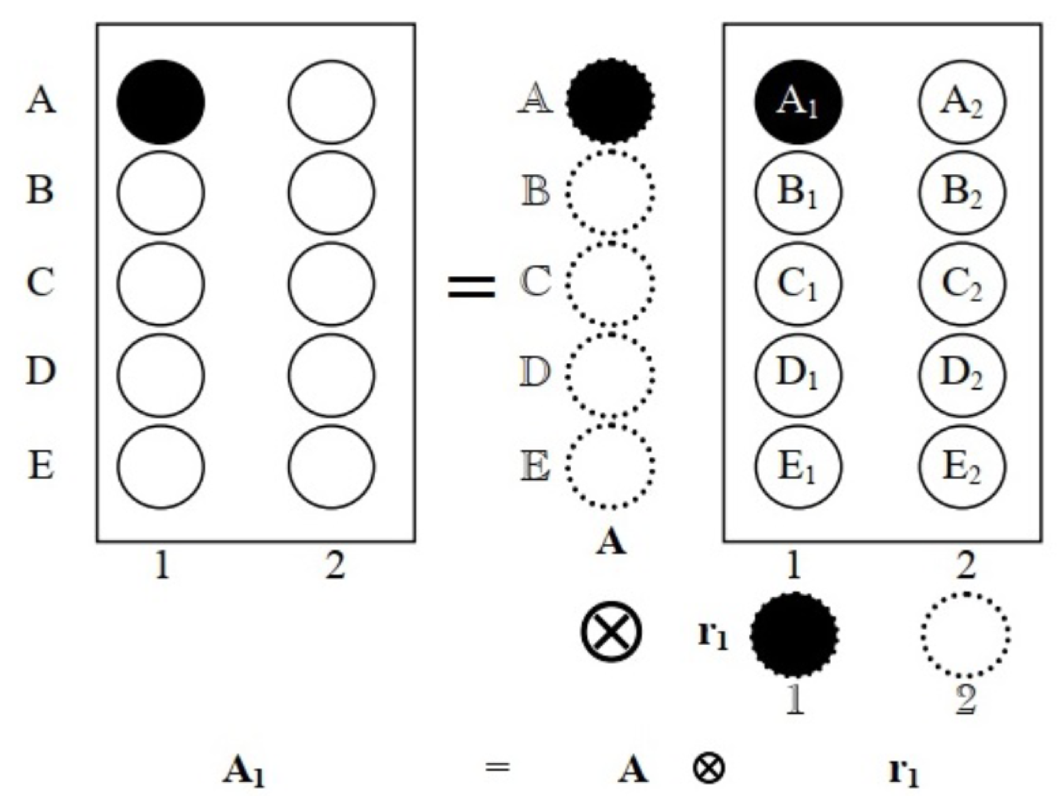
Overview

- Linguistic transductions $f : \Sigma^* \to \Gamma^*$ are bounded by the **regular functions** (Graf 2022; Heinz 2018)
- Constraint-interaction theories like OT (Eisner 1997,2000; Lamont 2019,2021), HG (Smolensky & Hale 1992), HS (Lamont 2019,2021; Hao 2021; Hampe 2022) do not respect this regular bound.
- One possible way out: direct embed regular constraints into the grammar using Tensor Product Representations (Smolensky 1990,2012)
- Rawski (2019): First-order (Star-Free) languages over arbitrary representations using tensor calculus.



Tensors as Functions

- **Order-k Tensor**: multi-way array, element of tensor product of k vector spaces
- **Tensor product** $\otimes$: generalization of the outer product of vector spaces
- **Tensor contraction** $\bullet$: generalization of the inner product: Order n $\bullet$ order m contraction yields tensor of order $n+m-2$ (sum through shared indices).

Tensor-multilinear map isomorphism:

For any multilinear map $f : V_1 \to \ldots \to V_n$ there is a tensor $T^f \in V_n \otimes \ldots \otimes V_1$ such that for any $\vec{v_1} \in V_1, \ldots, \vec{v}_{n-1} \in V_{n-1}$,

$$f(\vec{v_1}, \ldots, \overrightarrow{v_{n-1}}) = T^f \bullet \vec{v_1} \bullet \ldots \bullet \overrightarrow{v_{n-1}}$$

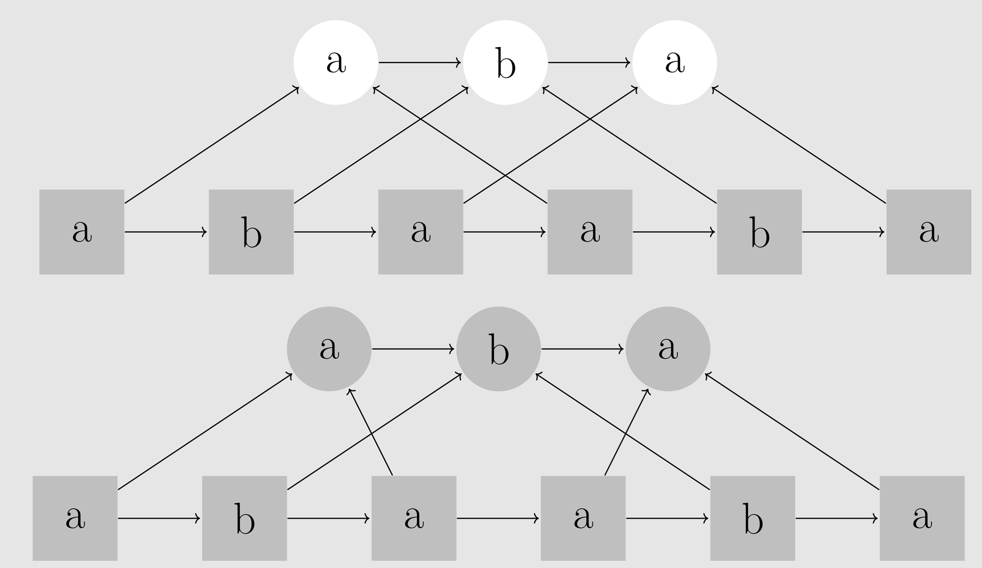
Tensors act as functions, with tensor contraction as function application.

Transductions as Origin Graphs

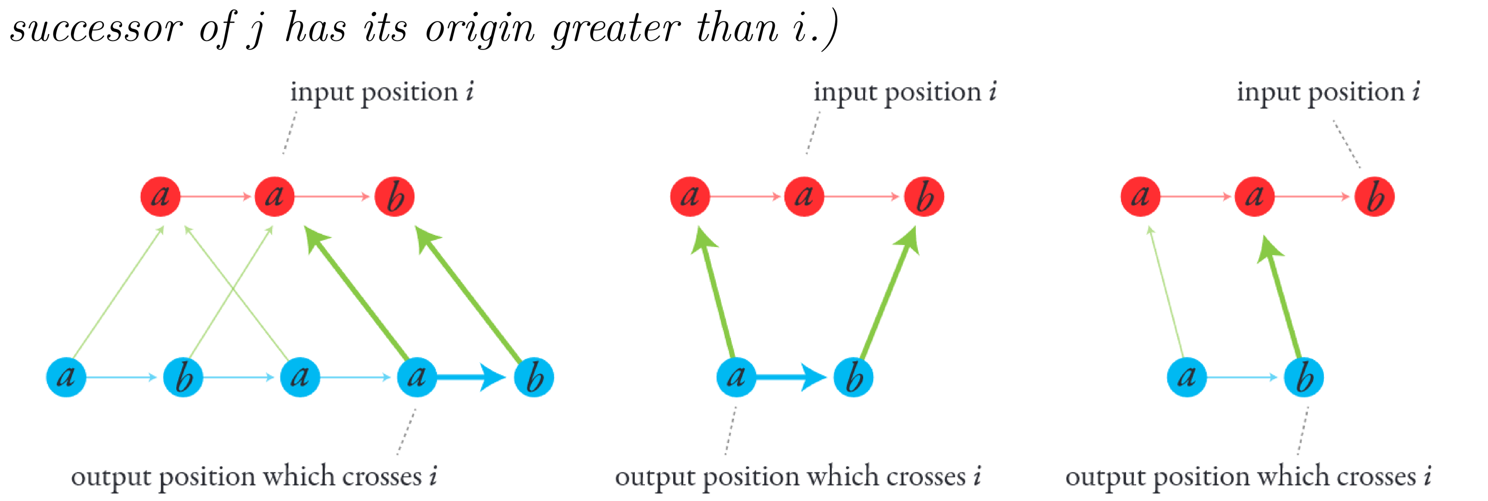
Proposal

View input-output mappings as (model-theoretic) structures, and mappings as (logical) well-formedness constraints on possible structures (like Correspondence Theory (Payne et al 2016))

Origin Graphs



Theorem 1. *(Bojańczyk et al., 2017) Let G be an origin transduction, i.e., an input alphabet, an output alphabet, and a set of origin graphs over these alphabets. G is a regular function (recognized by a streaming string transducer) iff:*
MSO-definable: *there is an MSO formula which is true in exactly the origin graphs from $\mathcal{G}$.*
bounded origin: *there is some $m \in \mathbb{N}$ such that in every origin graph from G, every input position is the origin of at most m output positions;*
k-crossing: *in every origin graph from $\mathcal{G}$, every input position is crossed by at most k output positions (An output position j crosses an input position i if the origin of j is no greater than i, and either j is the final output position, or the successor of j has its origin greater than i.)*



Tensor Representations of Origin Transductions

Embedding Origin Graphs via Finite Relational Models

- Domain elements D as the set of basis vectors $\mathcal{D} \cong \mathbb{R}^{|D|}$
- k-ary relation r computed by an order- k tensor $\mathcal{R}$
- Truth value $[\![r(d_{i_1}, \ldots, d_{i_k})]\!] = \mathcal{R}(\mathbf{d}_{i_1}, \ldots, \mathbf{d}_{i_k}) = \mathcal{R} \bullet \mathbf{d}_{i_1} \bullet \cdots \bullet \mathbf{d}_{i_k}$

Embedding Logical Connectives (Sato 2017):

$$[\![\neg F]\!]' = 1 - [\![F]\!]' \quad (1)$$
$$[\![F_1 \wedge \cdots \wedge F_h]\!]' = [\![F_1]\!]' \cdots [\![F_h]\!]' \quad (2)$$
$$[\![F_1 \vee \cdots \vee F_h]\!]' = \min_1([\![F_1]\!]' + \ldots + [\![F_h]\!]') \quad (3)$$
$$[\![\exists y F]\!]' = \min_1(\sum_{i=1}^{N} [\![F_{y \leftarrow e_i}]\!]') \quad (4)$$

Universal quantification over individual elements is treated as $\forall x F = \neg \exists x \neg F$.

$$[\![\exists X F]\!]' = \min_1 (\sum_{I \subseteq D} [\![F_{X \leftarrow I}]\!]') \quad (5)$$

Similarly, universal quantification over sets can be treated as $\forall X F = \neg \exists X \neg F$.

Properties in theorem 1

$$\text{Bounded origin} : \sum_{i=1}^{|M|} R_{\text{origin}}(x, i) \leq k \quad (6)$$

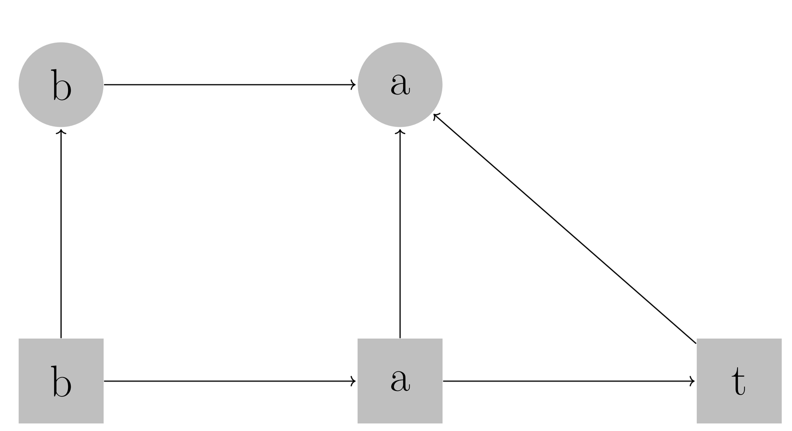
Binary relation R_{origin} defines the origin information between N and M. $R_{\text{origin}}(i, j) = 1$ when the output position j has the input position i as its origin.

$$k\text{-crossing} : \sum_{i=1}^{|M|} \text{cross}(x, i) \leq k \quad (7)$$

$\text{cross}(x, i)$ returns 1 if an output position i with origin at input position k preceding input position x, where i is either the last output position, or its successor has origin to the right of x.

Examples

-t insertion



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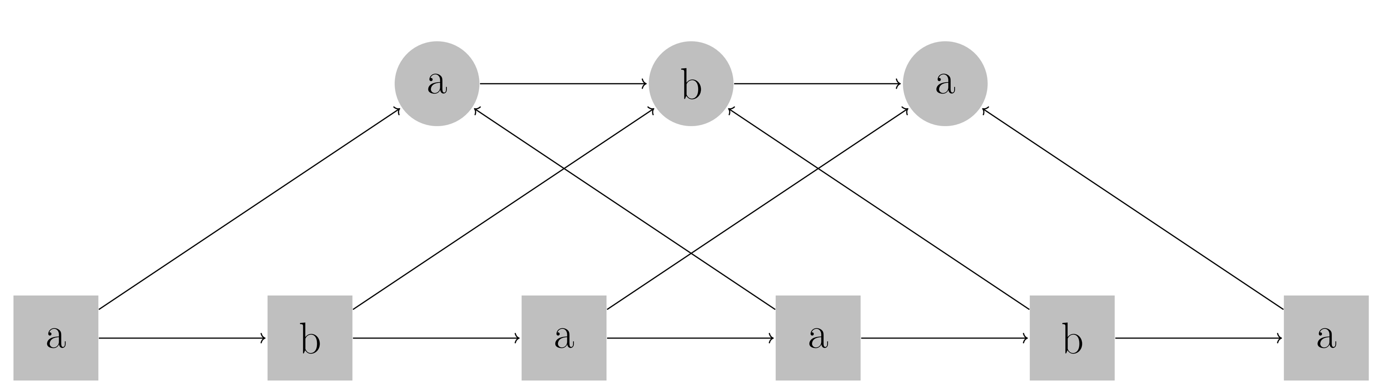
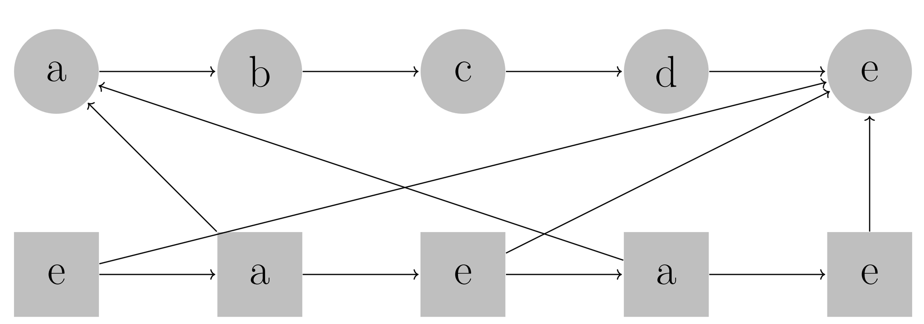


First-Last to Even-Odd mapping



Prenex normal form

$$F_{-t} = \forall x \exists y \exists z \exists x' \exists y' (\neg R_{\text{input}}(x) \vee (R_{\text{origin}}(x, y) \wedge R_{\text{equal}}(x, y) \wedge ((R_{\text{last-i}}(x) \wedge (R_{\text{succ-o}}(y, z) \wedge R_{\text{origin}}(x, z) \wedge R_{\text{t-o}}(z) \wedge R_{\text{last-o}}(z))) \vee (R_{\text{succ-i}}(x, x') \wedge R_{\text{succ-o}}(y, y') \wedge R_{\text{origin}}(x', y'))))) \quad (8)$$

Tensor notation

$$\mathcal{T}_{-t} = 1 - \min_1 \sum_{x=1}^{N} (1 - \min_1 \sum_{y,z,x',y'=1}^{N} (\min_1((1 - \mathcal{R}^{\text{input}} \mathbf{e}_x) + (\mathbf{e}_x^T \mathcal{R}^{\text{origin}} \mathbf{e}_y) \bullet (\mathbf{e}_x^T \mathcal{R}^{\text{equal}} \mathbf{e}_y) \bullet \min_1((\mathcal{R}^{\text{last-i}} \mathbf{e}_x) \bullet (\mathbf{e}_y^T \mathcal{R}^{\text{succ-o}} \mathbf{e}_z) \bullet (\mathbf{e}_x^T \mathcal{R}^{\text{origin}} \mathbf{e}_z) \bullet (\mathcal{R}^{\text{t-i}} \mathbf{e}_z) \bullet (\mathcal{R}^{\text{last-o}} \mathbf{e}_z) + ((\mathbf{e}_x^T \mathcal{R}^{\text{succ-i}} \mathbf{e}_{x'}) \bullet (\mathbf{e}_y^T \mathcal{R}^{\text{succ-o}} \mathbf{e}_{y'}) \bullet (\mathbf{e}_{x'}^T \mathcal{R}^{\text{origin}} \mathbf{e}_{y'})))))) \quad (9)$$